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INCREASE OF THE ELECTRON-POSITRON HAWKING RADIATION FROM SCHWARZSCHILD BLACK HOLES BY DIRAC MONOPOLES

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An algorithm for numerical computation of the barrier transparency for the potentials surrounding Schwarzschild black holes is described for massive spinor particles. It is then applied to calculate the total (including electronic neutrino and the contributions of twisted field configurations connected with Dirac monopoles) luminosity for the electron-positron Hawking radiation from a Schwarzschild black hole with mass $M = 10^{15}$ g. It is found that the contribution due to monopoles can be of order 12 % of the total electron-positron luminosity.

Keywords: Black holes; Hawking radiation; Dirac monopoles.

1. Introduction

The present paper is a natural continuation of our works [1,2] so, referring for more details on motivation of the given trend to the mentioned references, we shall here restrict ourselves to a short discussion.

Although it has elapsed over 25 years since Hawking discovered the possibility for black holes to radiate quantum particles, the situation with quantitative study of the Hawking radiation up to recently should be considered as extremely unsatisfactory. If qualitative understanding of the process developed more or less successfully then attempts of quantitative calculations might be enumerated by fingers. Really one may speak only about works of Refs.[3] and those of Refs.[4] based on Page's ones. However, when analysing the mentioned papers one can note that central problem of numerical calculations for Hawking radiation – calculating the barrier transparency for the potentials surrounding black holes – was actually not resolved so it is impossible to extract some exact algorithm for numerical computation of the necessary quantities from the mentioned references.

Another aspect of the problem in question is connected with that any isolated

black hole might possess the internal magnetic fields of the Dirac monopole types. The latter configurations should be connected with nontrivial topological properties of black holes and could have an essential influence on quantum processes near black holes, for instance, on Hawking radiation. A number of examples of such configurations may be found in Refs.[1] and references therein. Physically, the existence of those configurations should be obliged to the natural presence of magnetic U(N)-monopoles (with $N \geq 1$) on black holes though the total (internal) magnetic charge (abelian or nonabelian) of black hole remains equal to zero. One can consider that monopoles reside in black holes as quantum objects without having influence on the black hole metrics. They could reside in the form of monopole gas in which the process of permanent creation and annihilation of the virtual monopole-antimonopole pairs occurs so that the summed internal magnetic charge (i. e., related with topological properties) is equal to zero while the external one (not connected with topological properties) may differ from zero. While existing the virtual monopole-antimonopole pair can interact with a particle and, by this, increasing the Hawking radiation (see Refs.[1] and references therein).

In other words one may say that the black holes due to their nontrivial topological properties can actually carry the whole spectrum of topologically inequivalent configurations (TICs) for miscellaneous fields, in the first turn, complex scalar and spinor ones. The mentioned TICs can markedly modify the Hawking radiation from black holes. Physically, the existence of TICs should be obliged to the natural presence of magnetic U(N)-monopoles (with $N \geq 1$) on black holes and additional contributions to the Hawking radiation exist due to the additional scalar or spinor particles leaving the black hole because of the interaction with monopoles so the conforming radiation can be called *the monopole Hawking radiation*.[5]

Up to now, however, only influence of the TICs of complex scalar field on Hawking radiation has been studied more or less (see Refs.[1] and references therein for more details). The description of TICs for spinors was obtained in Refs.[5] but the detailed analysis of the TICs contribution to Hawking radiation requires knowledge of the conforming S-matrices which regulate the spinor particle passing through the potential barrier surrounding black hole. Those S-matrices for the Schwarzschild (SW) black holes have only recently been explored in Ref.[6] which allows us to obtain an algorithm to calculate the S-matrix elements numerically since for physical results to be obtained one needs to apply the numerical methods. In the massless case the S-matrices discussed are simpler to treat and the paper of Ref.[2] contained a description of the algorithm needed and applied it to calculate the all configurations luminosity for massless spinor particles for a SW black hole. The results obtained can serve as an estimate, in the first turn, of the electron-positron Hawking radiation and also for the neutrino one from the SW black holes. The more exact computation should take into account the particle masses and require more complicated algorithms for calculating the corresponding S-matrices.

The present paper contains a description of one such an algorithm and applies the latter to calculate the total luminosity for electrons, positrons, electronic neu-

trino and antineutrino for a SW black hole with mass $M = 10^{15}$ g. Section 2 is devoted to the common statement of the problem while Section 3 gives more detailed information about the relations needed for further calculations. Section 4 estimates the correctness of computations and Section 5 presents the numerical results obtained. Finally, Section 6 is devoted to concluding remarks.

We write down the black hole metric under discussion (using the ordinary set of local coordinates t, r, ϑ, φ) in the form

$$ds^2 = adt^2 - a^{-1}dr^2 - r^2(d\vartheta^2 + \sin^2\vartheta d\varphi^2) \tag{1}$$

with $a = 1 - 2M/r$ and M is the black hole mass.

Throughout the paper we employ the system of units with $\hbar = c = G = 1$, unless explicitly stated otherwise. We shall denote $L_2(F)$ the set of the modulo square integrable complex functions on any manifold F furnished with an integration measure while $L_2^n(F)$ will be the n-fold direct product of $L_2(F)$ endowed with the obvious scalar product.

Finally, when computing we use the following value of the electron-positron mass (see Ref.[7])

$$\mu_0(e^\pm) = 0.51099906 \text{ Mev} .$$

2. Preliminaries

As was disscussed in Refs.[5], TICs of a spinor field on black holes are conditioned by the availability of a countable number of the twisted spinor bundles over the $\mathbb{R}^2 \times \mathbb{S}^2$ -topology underlying the 4D black hole physics. From a physical point of view the appearance of spinor twisted configurations is linked with the natural presence of Dirac monopoles that play the role of connections in the complex line bundles corresponding to the twisted spinor bundles. Under the circumstances each TIC corresponds to sections of the corresponding spinor bundle E, which can be characterized by its Chern number $n \in \mathbb{Z}$ (the set of integers). Using the fact that all the mentioned bundles can be trivilized over the chart of local coordinates $(t, r, \vartheta, \varphi)$ covering almost the whole manifold $\mathbb{R}^2 \times \mathbb{S}^2$ one can obtain a suitable Dirac equation on the given chart for TIC Ψ with mass μ_0 and Chern number $n \in \mathbb{Z}$ that looks as follows

$$\mathcal{D}_n\Psi = \mu_0\Psi, \tag{2}$$

with the twisted Dirac operator $\mathcal{D}_n = i\gamma^\mu\nabla_\mu^n$ and we can call (standard) spinors corresponding to $n = 0$ *untwisted* while the rest of the spinors with $n \neq 0$ should be referred to as *twisted*. Referring for details and for explicit form of $\mathcal{D}_n$ to Refs.[5], it should be noted here that in $L_2^4(\mathbb{R}^2 \times \mathbb{S}^2)$ there is a basis from the solutions of (2) in the form

$$\Psi_{\lambda m} = \frac{1}{\sqrt{2\pi\omega}} e^{i\omega t} r^{-1} \begin{pmatrix} F_1(r, \omega, \lambda)\Phi_{\lambda m} \\ F_2(r, \omega, \lambda)\sigma_1\Phi_{\lambda m} \end{pmatrix} , \tag{3}$$

where σ_1 is the Pauli matrix, the 2D spinor $\Phi_{\lambda m} = \Phi_{\lambda m}(\vartheta, \varphi) = (\Phi_{1\lambda m}, \Phi_{2\lambda m})$ is the eigenspinor of the twisted euclidean Dirac operator with Chern number n on

the unit sphere with the eigenvalue $\lambda = \pm\sqrt{(l+1)^2 - n^2}$ while $-l \le m \le l+1$, $l \ge |n|$. As for the functions $F_{1,2}$, they obey the system of equations

$$\begin{cases} \sqrt{a}\partial_r F_1 + \left(\frac{1}{2}\frac{d\sqrt{a}}{dr} + \frac{\lambda}{r}\right) F_1 = i(\mu_0 - c)F_2, \\ \sqrt{a}\partial_r F_2 + \left(\frac{1}{2}\frac{d\sqrt{a}}{dr} - \frac{\lambda}{r}\right) F_2 = -i(\mu_0 + c)F_1 \end{cases} \tag{4}$$

with $c = \omega/\sqrt{a}$ and a of (1). The explicit form of the 2D spinor $\Phi_{\lambda m}$ is inessential in the given paper and can be found in Refs.[5]. One can only notice here that they can be subject to the normalization condition at n fixed

$$\int_0^\pi \int_0^{2\pi} (|\Phi_{1\lambda m}|^2 + |\Phi_{2\lambda m}|^2) \sin\vartheta d\vartheta d\varphi = 1$$

and these spinors form an orthonormal basis in $L_2^2(\mathbb{S}^2)$ at any $n \in \mathbb{Z}$.

By passing on to the Regge-Wheeler variable $r_* = r + 2M\ln(r/2M - 1)$ and by going to the quantities $x = r_*/M, y = r/M, k = \omega M, \mu = \mu_0 M$, we shall have $x = y + 2\ln(0.5y - 1)$, so that $y(x)$ is given implicitly by the latter relation (i.e., $-\infty < x < \infty$, $2 \le y < \infty$) with

$$y' = dy/dx = 1 - 2/y = (y-2)/y = a_0(x) \tag{5}$$

and the system (4) can be rewriten as follows

$$\begin{cases} E_1' + a_1 E_1 = b_1 E_2\ , \\ E_2' + a_2 E_2 = b_2 E_1 \end{cases} \tag{6}$$

with $E_1 = E_1(x,k,\lambda) = F_+(Mx), F_+(r^*) = F_1[r(r^*)], E_2 = E_2(x,k,\lambda) = iF_-(Mx), F_-(r^*) = F_2[r(r^*)]$ and

$$a_{1,2} = \frac{1}{2y^2} \pm \frac{\lambda}{y}\sqrt{a_0}\ , \tag{7}$$

$$b_{1,2} = \mu\sqrt{a_0} \mp k\ . \tag{8}$$

To evaluate luminosity of the Hawking radiation for spinor particles it is necessary to know the asymptotics of the functions $E_{1,2}$ at $x \to +\infty$. As was shown in Ref.[6], the latter asymptotics look as follows

$$E_1 \sim i\sqrt{k-\mu}\, s_{11}(k,\lambda)e^{ik^+x}e^{i\beta\ln(2k^+x)}, \qquad x \to +\infty\ , \tag{9}$$

$$E_2 \sim \sqrt{k+\mu}\, s_{11}(k,\lambda)e^{ik^+x}e^{i\beta\ln(2k^+x)}, \qquad x \to +\infty \tag{10}$$

with

$$\beta = \left(\frac{\mu}{k^+}\right)^2, \qquad k^+ = \sqrt{k^2 - \mu^2}\,,$$

where $s_{11}(k,\lambda)$ is an element of the S-matrix connected with some scattering problem for a Schrödinger-like equation (see Section 3). After this one can obtain (in

usual units) the luminosity $L(n)$ with respect to the Hawking radiation for TIC with the Chern number n in the form (for more details see Refs.[5,6])

$$L(n) = A\sum_{\pm\lambda}\sum_{l=|n|}^{\infty} 2(l+1)\int\limits_{\mu}^{\infty} \frac{|s_{11}(k,\lambda)|^2}{e^{8\pi k^+}+1}k^+dk =$$

$$A\sum_{l=|n|}^{\infty} 2(l+1)\int\limits_{\mu}^{\infty} \frac{|s_{11}(k,\lambda)|^2+|s_{11}(k,-\lambda)|^2}{e^{8\pi k^+}+1}k^+dk\,, \tag{11}$$

where $A = \frac{c^5}{\pi GM}\left(\frac{c\hbar}{G}\right)^{1/2} \approx 0.251455\cdot 10^{55}\,\mathrm{erg}\cdot \mathrm{s}^{-1}\cdot M^{-1}$ and M in g while $\mu = 3.762426569\cdot 10^{-18}\mu_0 M$ if μ_0 in MeV and M in g.

Luminosity $L(n)$ can be interpreted, as usual,[1,5] as an additional contribution to the Hawking radiation due to the additional spinor particles leaving the black hole because of the interaction with monopoles and, as already mentioned in Section 1, the conforming radiation can be called *the monopole Hawking radiation*.[5]

Under this situation, for the all configurations luminosity L of black hole with respect to the Hawking radiation concerning the spinor field to be obtained, one should sum up over all n, i. e.

$$L = \sum_{n\in\mathbb{Z}} L(n) = L(0) + 2\sum_{n=1}^{\infty} L(n) \tag{12}$$

since $L(-n) = L(n)$.

It should be emphasized that in (11), generally speaking, $s_{11}(k,\lambda) \neq s_{11}(k,-\lambda)$. Obviously, for neutrino there exists only $L(0)$ since neutrino does not interact with monopoles (when neglecting a possible insignificantly small magnetic moment of neutrino).

3. The scattering problem and description of algorithm

3.1. *The scattering problem*

It is evident that for numerical computation of all the above luminosities one needs to have some algorithm for calculating $s_{11}(k,\lambda)$. As was mentioned in Section 2, $s_{11}(k,\lambda)$ is an element of the S-matrix connected with some scattering problem for a Schrödinger-like equation. The latter equation, as was shown in Ref.[6], looks as follows

$$u'' + (k^2-\mu^2)u = q_0 u \tag{13}$$

with potential

$$q_0(x,k,\lambda,\mu) = \frac{\lambda^2\sqrt{a_0}}{y^2(x)}q(x,k,\lambda,\mu) + (\sqrt{a_0}-1)\mu^2 \tag{14}$$

while

$$q(x,k,\lambda,\mu) = \sqrt{a_0}\left(1-\frac{1}{y\lambda^2}\right) + \frac{y}{\lambda^2}q_1\left(\lambda+\frac{3y}{4\sqrt{a_0}}q_1\right) - \frac{y^2}{\lambda^2\sqrt{a_0}}\left(\frac{1}{2}q_2+q_3\right)$$

with

$$q_1 = -\frac{\mu\sqrt{a_0}}{y^2(k-\mu\sqrt{a_0})}\,, q_2 = -\frac{\mu\sqrt{a_0}(5/y-2)}{y^3(k-\mu\sqrt{a_0})}\,,$$
$$q_3 = -\frac{\lambda\sqrt{a_0}}{y^3}\left(y-3+\frac{\sqrt{a_0}}{\lambda}\right)\,.$$

As an example, in Figs. 1, 2 the numerically computed potentials $q_0(x,k,\lambda,\mu)$ with $\lambda > 0$ and $\lambda < 0$ are shown. It should be emphasized that potential q_0 of (14) satisfies the condition[6]

$$\int\limits_{-\infty}^{+\infty} |q_0|dx < \infty$$

only at $\mu = 0$, so when $\mu \neq 0$ the correct statement of the scattering problem for q_0 should be modified. Besides one can notice that when $\lambda > 0$, $q_0(x,\lambda) > 0$ at any $x \in [-\infty,\infty]$ while at $\lambda < 0$, $q_0(x,\lambda)$ can change the sign.

Figure 1: Typical potential barrier for $\lambda > 0$.

Figure 2: Typical potential barrier for $\lambda < 0$.

In its turn, as was shown in Ref.[6], the correct statement of the mentioned scattering problem for Eq. (13) consists in searching for two solutions $u^+(x,k,\lambda)$, $u^-(x,k,\lambda)$ of Eq. (13) obeying the following conditions

$$u^+(x,k,\lambda) = \begin{cases} e^{ikx} + s_{12}(k,\lambda)e^{-ikx} + o(1), & x\to-\infty, \\ s_{11}(k,\lambda)w_{i\beta,\frac{1}{2}}(-2ik^+x) + o(1), & x\to+\infty, \end{cases}$$

$$u^-(x,k,\lambda) = \begin{cases} s_{22}(k,\lambda)e^{-ikx} + o(1), & x\to-\infty, \\ w_{-i\beta,\frac{1}{2}}(2ik^+x) + s_{21}(k,\lambda)w_{i\beta,\frac{1}{2}}(-2ik^+x) + o(1), & x\to+\infty, \end{cases} \quad (15)$$

where the functions $w_{\pm i\beta,\frac{1}{2}}(\pm z)$ are related to the Whittaker functions $W_{\pm i\beta,\frac{1}{2}}(\pm z)$ (concerning the latter ones see e. g. Ref.[8]) by the relation

$$w_{\pm i\beta,\frac{1}{2}}(\pm z) = W_{\pm i\beta,\frac{1}{2}}(\pm z)e^{-\pi\beta/2}\,,$$

so that one can easily gain asymptotics (using the corresponding ones for Whittaker functions[8])

$$w_{i\beta,\frac{1}{2}}(-2ik^+x) = e^{ik^+x}e^{i\beta\ln|2k^+x|}[1+O(|k^+x|^{-1})],\; x\to+\infty\,,$$

$$w_{-i\beta,\frac{1}{2}}(2ik^+x) = e^{-ik^+x}e^{-i\beta\ln|2k^+x|}[1+O(|k^+x|^{-1})],\ x\to+\infty\,. \tag{16}$$

We can see that there arises some S-matrix with elements $s_{ij}, i,j = 1,2$. As has been seen above, for calculating the Hawking radiation we need the coefficient $s_{11}(k,\lambda)$, consequently, we need to have an algorithm for numerical computation of it inasmuch as the latter cannot be evaluated in exact form. The given algorithm can be extracted from the results of Ref.[6].

3.2. *Description of algorithm*

To be more precise

$$s_{11}(k,\lambda) = 2ik/[f^-(x,k,\lambda), f^+(x,k,\lambda)]\,, \tag{17}$$

where [,] signifies the Wronskian of functions f^-, f^+, the so-called Jost type solutions of Eq. (13). In their turn, these functions and their derivatives obey the certain integral equations. Since the Wronskian does not depend on x one can take the following form of the mentioned integral equations

$$f^-(x_0,k,\lambda) = e^{-ikx_0} + \frac{1}{k}\int\limits_{-\infty}^{x_0}\sin[k(x_0-t)]q^-(t,k,\lambda)f^-(t,k,\lambda)dt\,, \tag{18}$$

$$(f^-)'_x(x_0,k,\lambda) = -ike^{-ikx_0} + \int\limits_{-\infty}^{x_0}\cos[k(x_0-t)]q^-(t,k,\lambda)f^-(t,k,\lambda)dt\,, \tag{19}$$

and

$$f^+(x_0,k,\lambda) = w_{i\beta,\frac{1}{2}}(-2ik^+x_0) +$$

$$\frac{1}{k^+}\int\limits_{x_0}^{+\infty}\mathrm{Im}[w_{i\beta,\frac{1}{2}}(-2ik^+x_0)w_{-i\beta,\frac{1}{2}}(2ik^+t)]q^+(t,k,\lambda)f^+(t,k,\lambda)dt\,, \tag{20}$$

$$(f^+)'_x(x_0,k,\lambda) = \frac{d}{dx}w_{i\beta,\frac{1}{2}}(-2ik^+x_0) +$$

$$\frac{1}{k^+}\int\limits_{x_0}^{+\infty}\mathrm{Im}\left[\frac{d}{dx}w_{i\beta,\frac{1}{2}}(-2ik^+x_0)w_{-i\beta,\frac{1}{2}}(2ik^+t)\right]q^+(t,k,\lambda)f^+(t,k,\lambda)dt\,, \tag{21}$$

where

$$q^-(x,k,\lambda) = \frac{\lambda^2\sqrt{a_0}}{y^2}q + a_0\mu^2\,, q^+(x,k,\lambda) = \frac{\lambda^2\sqrt{a_0}}{y^2}q + 2\mu^2\frac{y-x}{xy} \tag{22}$$

with $q = q(x,k,\lambda,\mu)$ from (14).

The potential q^- exponentially tends to zero when $x\to-\infty$ and potential q^+ behaves as $O(x^{-2})$ as $x\to+\infty$. So one can notice that these potentials are integrable when $x\to-\infty$ or $x\to\infty$ respectively. The point $x = x_0$ should be

chosen from the considerations of the computational convenience. The relations (18)–(22) can be employed for numerical calculation of s_{11}. It should be noted, however, that while Eqs. (18)–(19) are suitable for direct calculation this is not the case for Eqs. (20)–(21) — it is difficult to evaluate the functions $w(z)$ in (20)–(21) since we have yet no effective fast method to compute the Whittaker functions $W(z)$ at z required. In the present paper we, therefore, stick to the following strategy. We employ the asymptotic expressions of w-functions at $x \to +\infty$ from Eq. (16) and replace Eqs. (20)–(21) by

$$f^+(x_0,k,\lambda) = e^{i(k^+x+\mu\ln(2k^+x_0))} + \frac{1}{k^+}\int\limits_{x_0}^{+\infty} \mathrm{Im}[e^{i(k^+(x_0-t)+\mu\ln(x_0/t))}]q^+(t,k,\lambda)f^+(t,k,\lambda)dt\,, \tag{23}$$

$$(f^+)'_x(x_0,k,\lambda) = ie^{i(k^+x_0+\mu\ln(2k^+x_0))}(k^+ + \mu/x_0) + \int\limits_{x_0}^{+\infty} \mathrm{Im}\left[i\left(1+\frac{\mu}{k^+x_0}\right)e^{i(k^+(x_0-t)+\mu\ln(x_0/t))}\right]q^+(t,k,\lambda)f^+(t,k,\lambda)dt\,. \tag{24}$$

The latter equations are appropriate for numerical evaluation but under this approximation, as numerical experiment shows, reliable calculations are only possible for $\mu < 0.01$. This upper limit corresponds to $\mu_0(e^\pm)$ and $M \sim 10^{15}$ g and it is obtained when requiring that calculations lead to the smooth monotonous graphs for $\Gamma(k,\pm\lambda) = |s_{11}(k,\pm\lambda)|^2$ without any oscillations, breaks or fractures (see below Figs. 3,4). In its turn, the latter requirement is based on the theoretical results that $\Gamma(k,\pm\lambda)$ is a smooth monotonous function of k changing between 0 and 1 (see Ref.[6]). Therefore in the present paper we restrict ourselves to considering electrons and positrons for $M = 10^{15}$ g. Including the heavier fundamental fermions ($\mu^\pm$-mesons and $\tau^\pm$-leptons) within the given algorithm is only possible for smaller M.

4. Estimations for luminosities

Before presentation of the numerical results we should touch upon the convergence of the series (11)–(12) over l and n respectively. For this aim we denote

$$c_l(n) = \int\limits_{\mu}^{\infty} \frac{|s_{11}(k,\lambda)|^2 + |s_{11}(k,-\lambda)|^2}{e^{8\pi k^+}+1}k^+dk$$

and represent the coefficients $c_l(n)$ in the form (omitting the integrand)

$$c_l(n) = c_{l1}(n) + c_{l2}(n) = \int\limits_{\mu}^{(\ln l)/2\pi} + \int\limits_{(\ln l)/2\pi}^{\infty}\,, \tag{25}$$

so that $L(n)$ of (11) is equal to $L_1(n)+L_2(n)$ respectively. Also it should be noted, as follows from the results of Ref.[6], we have for barrier transparency $\Gamma(k,\pm\lambda) = |s_{11}(k,\pm\lambda|^2$ that $\Gamma(\mu,\pm\lambda)=0$ and at $k\to+\infty$

$$\Gamma(k,\pm\lambda) = |s_{11}(k,\pm\lambda)|^2 = 1+O(k^{-1})\,. \tag{26}$$

Further we have the asymptotic behaviour for large l and $k << l$

$$|s_{11}(k,\pm\lambda)| = C\frac{e^{-\sqrt{n+l+1}}}{\sqrt{n+l+1}}(k-\mu)[1+o(1)] \tag{27}$$

with some constant C. To obtain it we should work using the equations (18)–(21) by conventional methods of mathematical analysis so that the detailed derivation of (27) lies somewhat out of the scope of the present paper and we will not dwell upon it here. Using (25) and (27) we find

$$c_{l1}(n) \le B\frac{e^{-2\sqrt{n+l+1}}}{n+l+1}\int_{\mu}^{(\ln l)/2\pi}\frac{(k-\mu)^2k^+}{e^{8\pi k^+}+1}dk$$

with some constant B and then it is clear that

$$c_{l1}(n) \le B\frac{e^{-2\sqrt{n+l+1}}}{n+l+1}\int_{0}^{(\ln l)/2\pi}\frac{(k-\mu)^2k^+}{e^{8\pi k^+}-1}dk$$

$$\le B\frac{e^{-2\sqrt{n+l+1}}}{n+l+1}\int_0^\infty\frac{(k^+)^3}{e^{8\pi k^+}-1}dk^+ = B\frac{3!\zeta(4)}{(8\pi)^4}\frac{e^{-2\sqrt{n+l+1}}}{n+l+1}\,,$$

where we used the formula (for natural $p>0$)[8]

$$\int\limits_0^\infty \frac{t^p dt}{e^t-1} = p!\,\zeta(p+1)$$

with the Riemann zeta function $\zeta(s)$, while $\zeta(4)=\pi^4/90$. As a consequence, one may consider

$$L_1(n) \sim \int\limits_n^\infty 2(l+1)\frac{e^{-2\sqrt{n+l+1}}}{n+l+1}dl \sim 2\int\limits_{\sqrt{2n+1}}^\infty e^{-2t}\left(t-\frac{n}{t}\right)dt$$

$$\sim e^{-2\sqrt{2n+1}}\left(\sqrt{2n+1}-\frac{1}{\sqrt{2n+1}}\right)\,, \tag{28}$$

where we employed the asymptotical behaviour of the incomplete gamma function[8]

$$\Gamma(\alpha,x) = \int\limits_x^\infty e^{-t}t^{\alpha-1}dt \sim x^{\alpha-1}e^{-x},\; x\to+\infty\,,$$

so that the series $\sum_1^\infty L_1(n)$ is evidently convergent.

At the same time due to (26)

$$c_{l2}(n) \leq D \int\limits_{(\ln l)/2\pi}^{\infty} e^{-8\pi k^+} k^+ dk =$$

$$D \int\limits_{z}^{\infty} e^{-8\pi k^+} \frac{(k^+)^2}{k} dk^+ \ , \ z = \sqrt{(\ln l/2\pi)^2 - \mu^2}$$

with some constant D. Since $k^+/k < 1$ and

$$\int\limits_{(\ln l)/2\pi}^{\infty} e^{-8\pi k^+} k^+ dk^+ \sim \frac{\ln l}{l^4}$$

then

$$c_{l2}(n) \leq D \frac{\ln l}{l^4} \ . \tag{29}$$

This entails

$$L_2(n) \sim \int\limits_{n}^{\infty} \frac{2(l+1)\ln l}{l^4} dl \sim \frac{\ln n}{n^2} \ . \tag{30}$$

and the series $\sum_1^\infty L_2(n)$ is also convergent. Under this situation we obtain that the series of (12) is convergent, i. e., $L < +\infty$. Consequently, we can say that all the further computed luminosities exist and are well defined.

5. Numerical results

In view of (26), we have

$$\int\limits_{\mu}^{\infty} \frac{\Gamma(k,\lambda)k^+ dk}{e^{8\pi k^+}+1} \sim \int\limits_{0}^{\infty} \frac{kdk}{e^{8\pi k}-1} = \frac{1}{(8\pi)^2}\zeta(2) = \frac{1}{6\cdot 8^2} \ , \tag{31}$$

whilst the latter integral can be accurately evaluated using the trapezium formula on the [0, 3] interval. Having confined the range of k to the [μ, 3] interval, accordingly, it is actually enough to restrict oneself to $0 \leq l, n \leq$15–20 when computing $L(n), L$. We took $0 \leq l \leq 20$, $0 \leq n \leq 15$. Since the Wronskian of (17) does not depend on x, the latter should be chosen in the region where the potentials q^+ of (22) are already small enough. Besides, potentials q^- of (22) are really equal to 0 when $x <$ -20. We computed $\Gamma(k,\lambda)$ according to (17) at $x = x_0 = 300$, where $f^\pm$ were obtained from the Volterra integral equations, respectively, (18) and (23). For this aim, according to the methods developed for numerical solutions of integral equations (see e. g., Ref.[9]), those of (18) and (23) have been replaced by systems of linear algebraic equations which can be gained when calculating the conforming integrals by the trapezium formula, respectively, for the [-20, 300] and [300, 400] intervals. The

sought values of $f^{\pm}$ were obtained as the solutions to the above linear systems. After this the derivatives $(f^{\pm})'_x$ were evaluated in accordance with (19) and (24) while employing the values obtained for $f^{\pm}$. The typical behaviour of $\Gamma(k,\lambda)$ is presented in Figs. 3, 4.

Figure 3: Typical behavior of barrier transparencies for $\lambda > 0$.

Figure 4: Typical behavior of barrier transparencies for $\lambda < 0$.

Further Fig. 5 presents luminosity $L_\nu(0)/A$ with A of (11) for electronic neutrino (antineutrino) (no interaction with monopoles) as function of k and also the untwisted $L_e(0)/A$ and the all configurations L_e/A luminosities for electron (positron) as functions of k. The areas under the curves give the corresponding values of $L(0)/A$ and L/A.

Figure 5: Luminosity for electronic neutrino (antineutrino) and untwisted and all configurations luminosities for electron (positron).

In Table 1 the data on computation of the untwisted and the all configurations luminosities are represented for all particle species under consideration. Finally, we can introduce the *total* untwisted $\mathcal{L}_0$ (as the sum over all species in the second column of Table 1) and the *total* $\mathcal{L}$ (as the sum over all species in the third column of Table 1) luminosities multiplied by A that should be detected by an external observer near black hole. We shall have

$$\mathcal{L}_0 = 0.100317 \cdot 10^{38}\,\mathrm{erg} \cdot \mathrm{s}^{-1},\ \mathcal{L} = 0.114389 \cdot 10^{38}\,\mathrm{erg} \cdot \mathrm{s}^{-1}$$

so that contribution owing to Dirac monopoles amounts to 12.3022 % of $\mathcal{L}$.

6. Concluding remarks

It seems to us the considerations of the present paper show that exact quantitative study of the Hawking radiation is not so simple task. At the same time such an exploration is important enough in a number of astrophysical problems, for example, for more precise determination of the masses of primordial black holes which just expire today. But, as we have seen, just the black hole mass is one of the crucial parameters that define both the relevant scattering problems and the algorithms necessary for numerical calculations. Besides the exact evaluations might shed a light on nontrivial topological properties of black holes, in particular, on those connected with residence of Dirac monopoles (and generally U(N)-monopoles with $N > 1$) in black holes. As a result, one should continue the exact calculations

Table 1. Untwisted and the all configurations luminosities.

Particle	$L(0)/A$	L/A	Monopole contribution of L (%)
ν_e	$0.786745{\cdot}10^{-3}$	$0.786745{\cdot}10^{-3}$	0
$\tilde{\nu_e}$	$0.786745{\cdot}10^{-3}$	$0.786745{\cdot}10^{-3}$	0
e^-	$0.120798{\cdot}10^{-2}$	$0.148780{\cdot}10^{-2}$	18.8072
e^+	$0.120798{\cdot}10^{-2}$	$0.148780{\cdot}10^{-2}$	18.8072

within the broader spectrum of both black hole masses and quantum particles emitting from black holes, in first turn, for other fundamental fermions – $\mu^{\pm}$-mesons and $\tau^{\pm}$-leptons. Also one should pass on to the other types of black holes, for instance, to the Reissner-Nordström ones where the description of the necessary S-matrices for spinor particles has been just recently obtained.[10]

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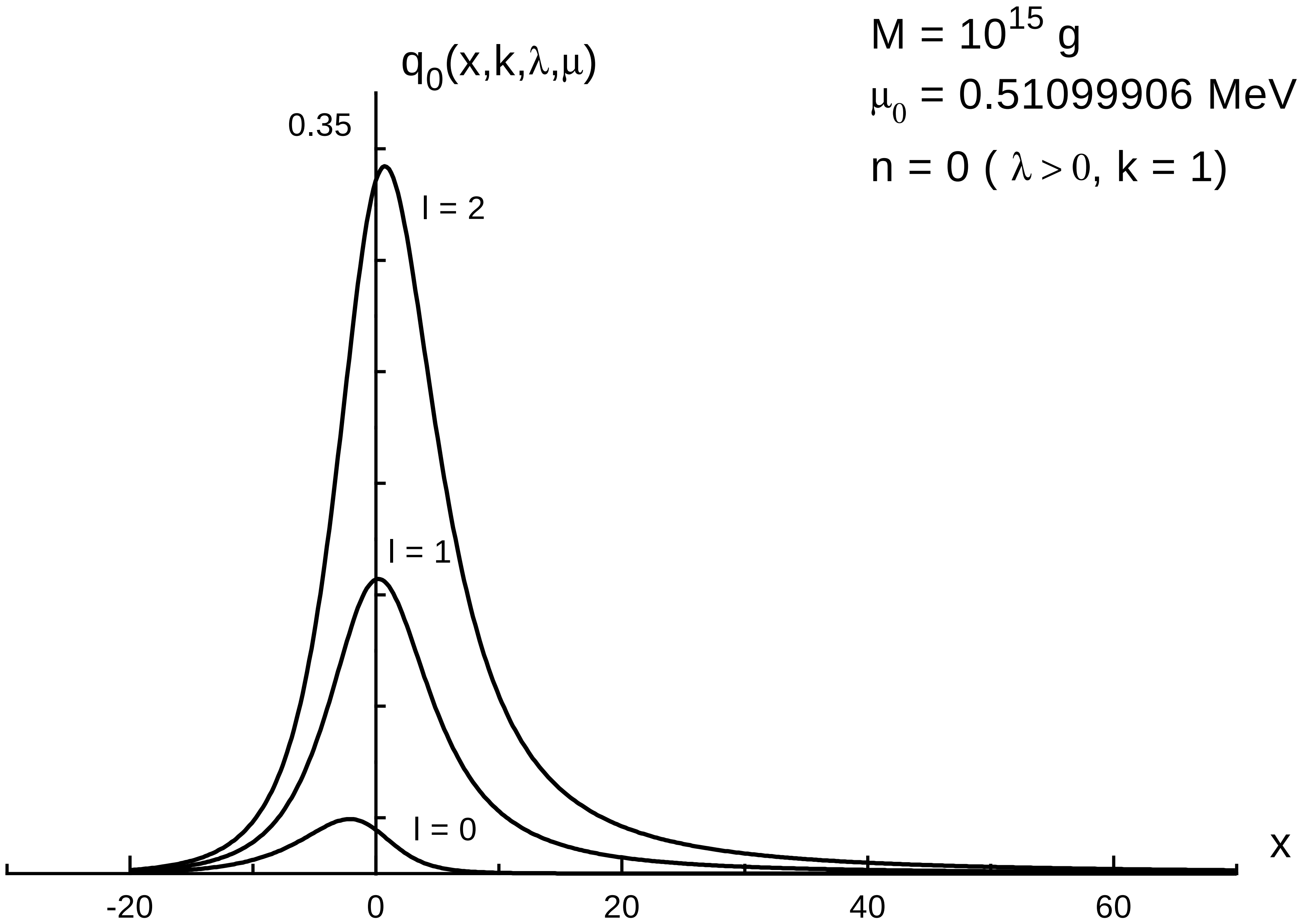
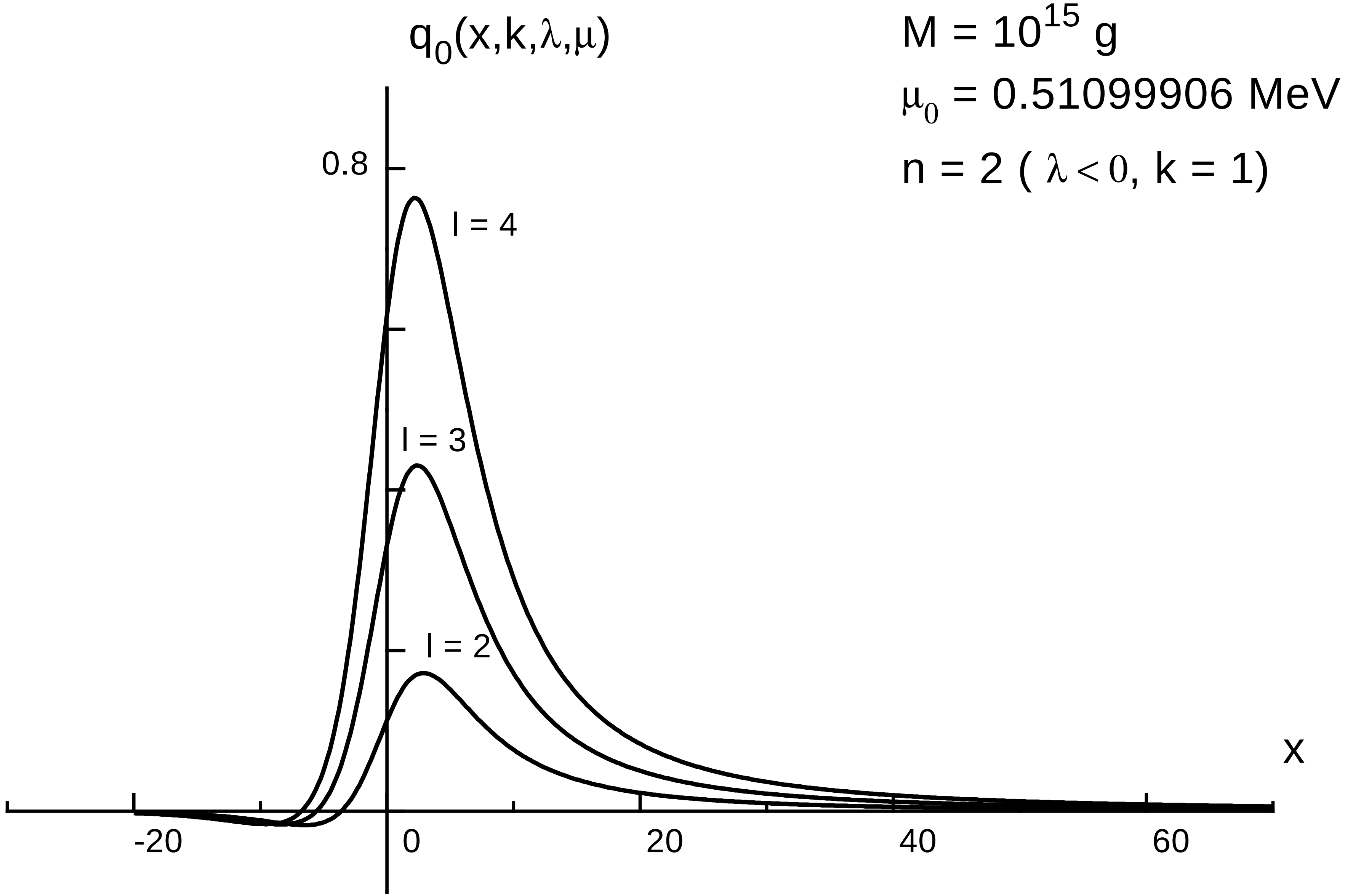
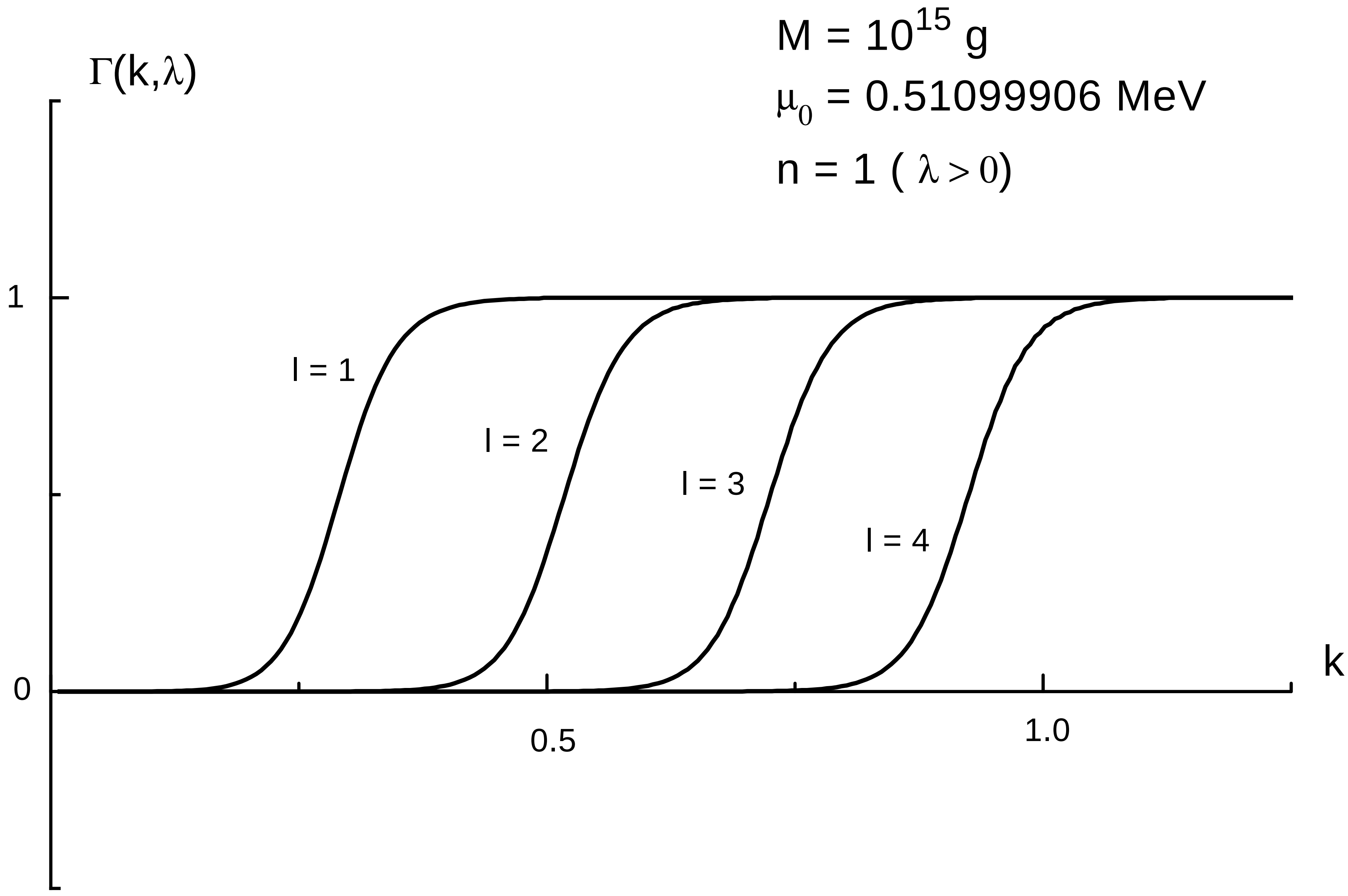

$q_0(x,k,\lambda,\mu)$
0.35
$M = 10^{15}$ g
$\mu_0 = 0.51099906$ MeV
$n = 0$ ($\lambda > 0$, $k = 1$)
l = 2
l = 1
l = 0
x
-20
0
20
40
60

Fig. 1



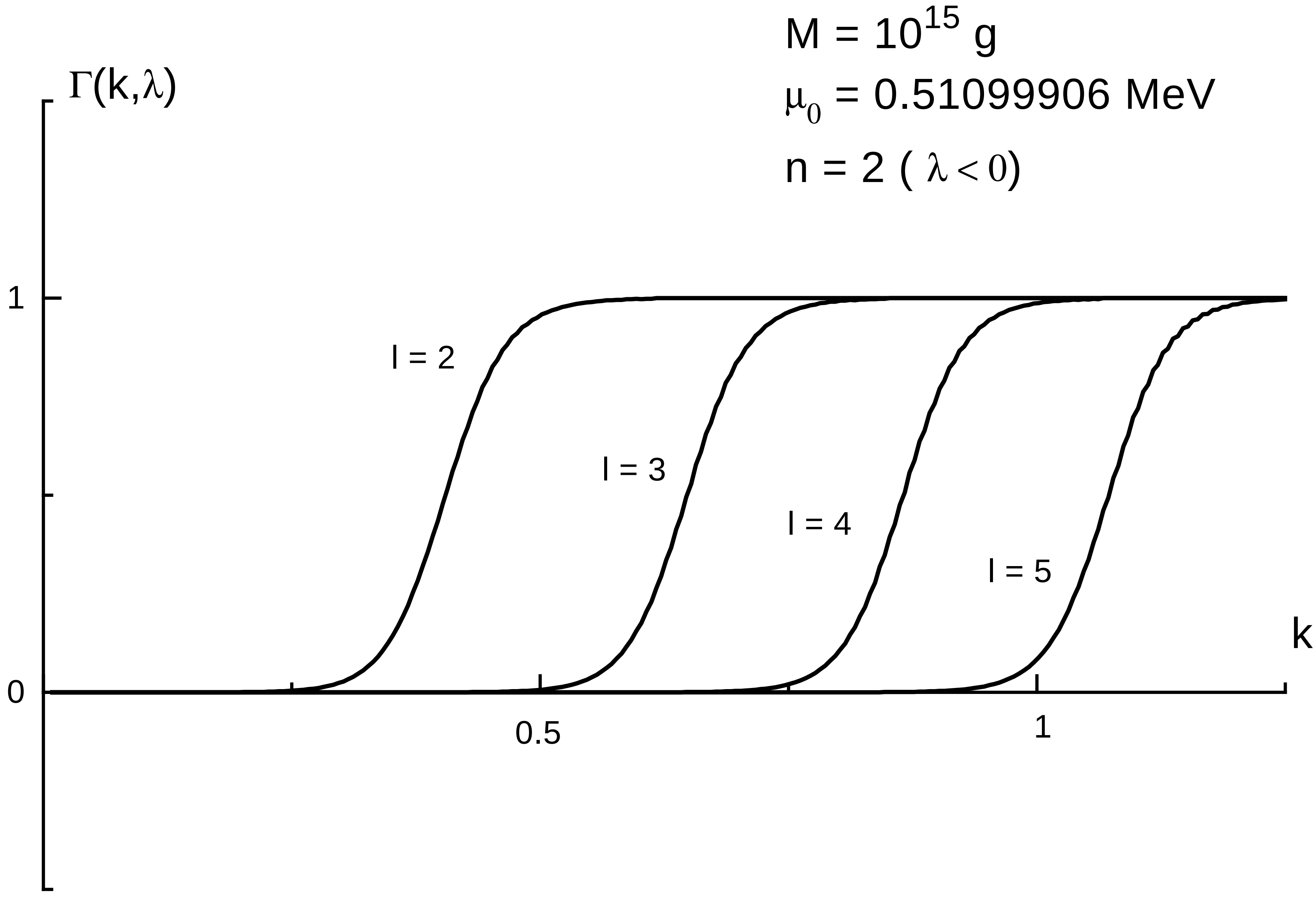
$q_0(x,k,\lambda,\mu)$
$M = 10^{15}$ g
μ_0 = 0.51099906 MeV
n = 2 ($\lambda < 0$, k = 1)
0.8
l = 4
l = 3
l = 2
x
-20
0
20
40
60

Fig. 2



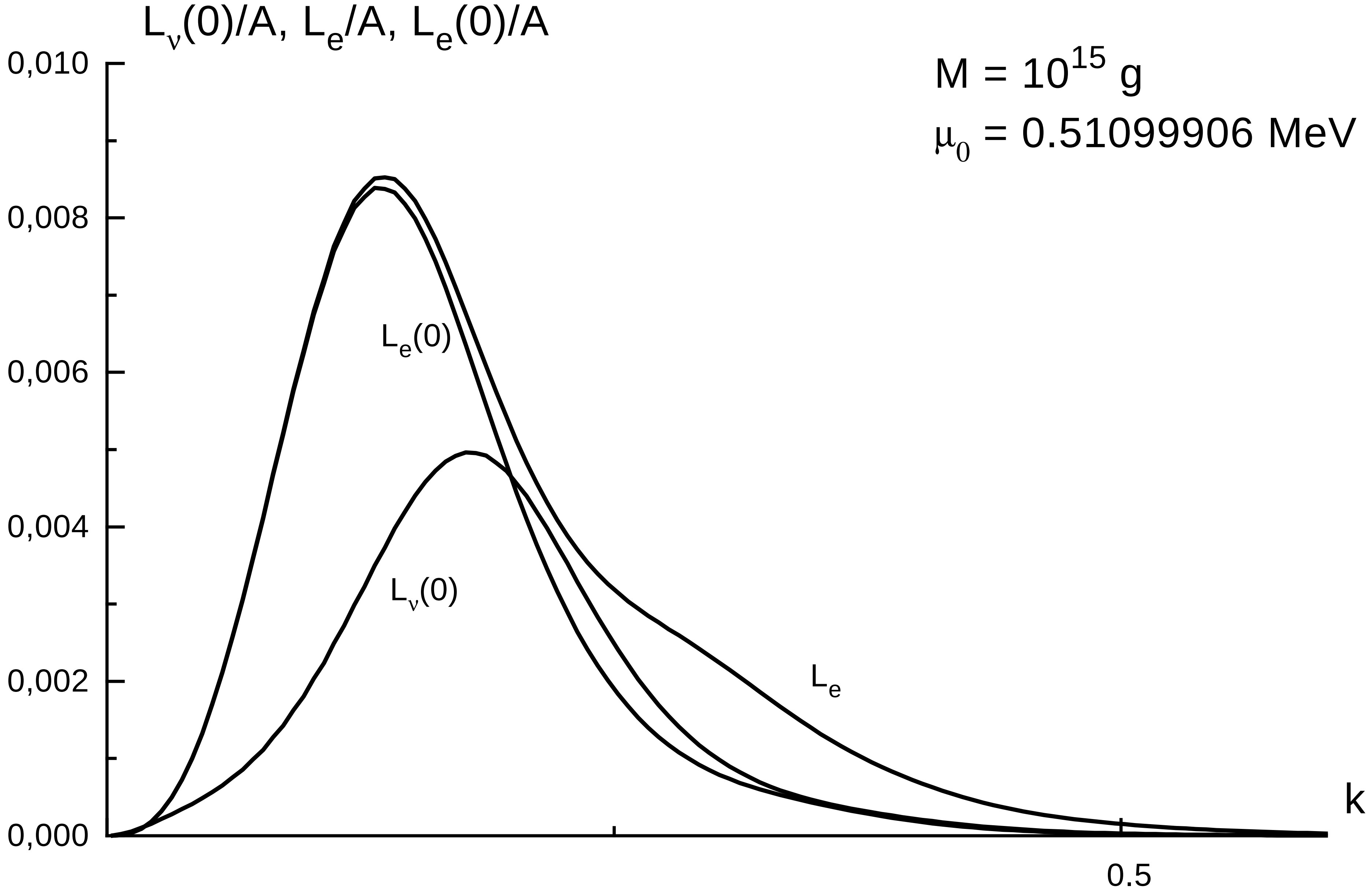
$M = 10^{15}$ g
$\mu_0 = 0.51099906$ MeV
n = 1 ($\lambda > 0$)
$\Gamma(k,\lambda)$
1
0
l = 1
l = 2
l = 3
l = 4
k
0.5
1.0

Fig. 3



M = 10^{15} g
μ_0 = 0.51099906 MeV
n = 2 ($\lambda < 0$)
$\Gamma(k,\lambda)$
1
0
0.5
1
k
l = 2
l = 3
l = 4
l = 5

Fig. 4



$L_\nu(0)/A$, L_e/A, $L_e(0)/A$
0,010
0,008
0,006
0,004
0,002
0,000
$M = 10^{15}$ g
$\mu_0 = 0.51099906$ MeV
$L_e(0)$
$L_\nu(0)$
L_e
k
0.5

Fig. 5